<u>Poisson Distribution</u>

$$P(X = x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

This is the probability density function of the Poisson distribution. $P(X = x)$ tells us we are dealing with a probability when x is a certain value. $e$ is a mathematical constant (a number with a fixed value, this value is 2.71828…). $\mu$ is the mean (the average). $x$ is the number of times the event occurs

The other two distributions that we have covered are the binomial and normal distributions. The former is for instances were there can only be success or failure (e.g. *"Seeds have a probability of germinating of 0.9. If six seeds are sown what is the probability of five seeds germinating"* would be a typical Binomial question (germinating or not germinating are the success or failure), the latter is for continuous[1] distributions (e.g. *"An industrial process mass produces an item whose weights are normally distributed with mean 18.5kg and standard deviation[2] 1.5kg. What is the probability that an item chosen at random weighs 21.5kg?"* would be a typical Normal question).

There are two primary purposes of the Poisson distribution:

1. Estimation of the probability of random events which have a small probability of happening.
2. Estimation of the binomial distribution, when there is a large number of trials and a low chance of success e.g. more than 50 seeds and an individual probability of germinating is less than 0.1.

For example, telephone calls arrive at a call centre on at the rate of 50 per hour. We want to find the probabilities of 0, 1 or 2 calls arriving in any 6 minute period.

First find the average rate of calls for the 6 minute period. 6 minutes is 1/10th of an hour, so there will be on average 5 calls per 6 minute period.

$$P(X = 0) = \frac{e^{-5}5^{0}}{0!}$$

For 0 calls, the probability can be calculated to be 0.0067…

Rather than carry out this process again for 1 and 2, as this is quite a laborious calculation, we can use the Poisson recurrence formula:

$$P(X = x + 1) = \frac{\mu}{x + 1} \times P(X = x)$$

So, for 1 call in 6 minutes:

$$P(X = 0 + 1) = \frac{5}{0 + 1} \times P(X = 0)$$

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[1] relating to items that can have any value within a given range (e.g. height of a person can be 1.66m, 2.0956m, or any other value to as many or as few decimal places) the opposite of this is something that is discrete, and can only take specific values (e.g. number of sweets in a jar, can only be 0,1,2 etc. – you cannot have 1.66 sweets or 2.0956 sweets)

[2] this is a measure of the average difference of a value from the mean. In an exam this will probably be a small single-digit value.

$$P(X=1)=\frac{5}{1}\times 0.0067\ldots$$
$$P(X=1)=0.0336\ldots$$

For 2 minutes:

$$P(X=1+1)=\frac{5}{1+1}\times P(X=1)$$
$$P(X=2)=\frac{5}{2}\times 0.0336\ldots$$
$$P(X=2)=0.0842\ldots$$

If we add these probabilities we find that the chance that $P(X \geq 2)$ is therefore 0.1246… (about 12.5%).

Proof

As is often the case, this is not exactly a proof, but shows why this recurrence relation works.

$$P(X=0)=\frac{e^{-5}5^0}{0!}=e^{-5}$$
$$P(X=1)=\frac{e^{-5}5^1}{1!}=5e^{-5}=5P(X=0)$$
$$P(X=2)=\frac{e^{-5}5^2}{2!}=\frac{25}{2}e^{-5}=\frac{5}{2}P(X=1)$$
$$P(X=3)=\frac{e^{-5}5^3}{3!}=\frac{125}{6}e^{-5}=\frac{5}{3}P(X=2)$$

It is clear the equation can be used

$$P(X=x+1)=\frac{\mu}{x+1}\times P(X=x)$$

See also
- Binomial Distribution
- Normal Distribution

References
Graham, C., Graham, D. and Whitcombe, A. (1986). *A-level Mathematics*. London: Charles Letts. p.152.